SOME RESULTS ON THE INTEGRABILITY OF EINSTEIN'S FIELD EQUATIONS FOR AXISTATIONARY PERFECT FLUIDS

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Using an orthonormal Lorentz frame approach to axistationary perfect fluid spacetimes, we have formulated the necessary and sufficient equations as a first order system, and investigated the integrability conditions of this set of equations. The integrability conditions are helpful tools when it comes to check the consequences and/or compatibility of certain simplifying assumptions, e.g. Petrov types. Furthermore, using this method, a relation between the fluid shear and vorticity is found for barotropic fluids. We collect some results concerning Petrov types, and it is found that an incompressible axistationary perfect fluid must be of Petrov type I.

We are interested in axistationary perfect fluid spacetimes, i.e. the energy-momentum tensor takes the form $T_{ab} = (\mu + p)u_a u_b - p g_{ab}$, and the indices $a, b, ...$ refere to a Lorentz frame comoving with the fluid (here p is pressure, μ density). Furthermore the spacetime has two commuting Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$, and all our quantities depend only on the coordinates x and y. The frame $\{e_a\}$ is adapted to the spacetime symmetry, i.e., $e_1 = \xi_1\partial_x + \upsilon_1\partial_y$, $e_2 = \xi_2\partial_x + \upsilon_2\partial_y$, while e_0 and e_3 do not contain ∂_x or ∂_y. With these choices, we bring the Riemann tensor and the Ricci rotation coefficients to "standard form" for axistationary perfect fluids.[1] Inherent in this standard form is that we still have the freedom to perform a rotation in the tetrad plane spanned by e_1 and e_2, which may be used to reduce the number of variables, and thus simplify the equations.[1]

Our variables – the Riemann tensor, the Ricci rotation coefficients, and the tetrad vector components – are constrained by (a) the Bianchi identities, (b) the Ricci identities, and (c) the commutator equations $[e_1, e_2] = 2\Gamma^a{}_{[21]}e_a$, $a = 1, 2$, respectively ($\Gamma^a{}_{bc}$ being the Ricci rotation coefficients). These equations form a set equivalent to Einstein's equations when the Ricci tensor is related to the energy-momentum tensor.[1,2,3]

The procedure of checking integrability of this first order system is to apply the commutator to equations (a) and (b). We know that (and this is straightforward to check) the Bianchi identities are the integrability conditions of the Ricci identities. Thus, in order to guarantee integrability, we apply the commutator to the Bianchi identites. The integrability condition of the momentum conservation equations $e_a(p) = \dot{u}_a(\mu + p)$, $\dot{u}_a$ being the fluid acceleration – is

$$\dot{u}_1 e_2(\mu) - \dot{u}_2 e_1(\mu) = 4(\mu + p)(\omega_1\sigma_2 - \omega_2\sigma_1) \; , \tag{1}$$

ω_a and σ_a being components of the fluid vorticity and shear, respectively (for the

definitions, see Ref. 1). It is known that rigid fluids (i.e. $\sigma_1 = \sigma_2 = 0$) must have a barotropic equation of state,[4] but using (1) we can state a more general result:

Theorem 1 *The equation of state is barotropic if and only if* $\omega_1\sigma_2 = \omega_2\sigma_1$.

The investigation of the integrability of the system of equations shows that the four Bianchi identities are satisfied due to the Ricci identities. There remain 4 equations (from the Bianchi identities) for four components of the Weyl tensor, and the two momentum conservation equations for the pressure, and these equations will have solutions according to the Cauchy–Kowalewski theorem.[5]

Next we list some results concerning some special subcases.

• Perfect fluids with purely magnetic Weyl tensor[1] can be investigated in the special cases (a) incompressible fluid, when the Bianchi identities imply conformal flatness, and by a theorem due to Collinson[6] this is the interior Schwarzschild solution, and (b) rigidly rotating fluid (i.e., shear-free), for which all solutions exhibit local rotational symmetry[7] which means that they either are the interior Schwarzschild solution, or NUT-like fluids.[1,8]

• Petrov types can be investigated when our orthonormal frame is related to a null tetrad, which will give the Weyl spinor in terms of the magnetic and electric parts of the Weyl tensor. (a) Petrov type III is inconsistent with the axistationary assumption,[1] (b) Petrov type N turns out to be equivalent to a vacuum spacetime with a cosmological constant $\Lambda = -p$;[1] (c) using the perturbative approach of Hartle[9] combined with the tetrad formalism, Fodor & Perjés[10] have shown that (i) Petrov type II reduce to the de Sitter spacetime in the static limit, and (ii) Petrov type D cannot be incompressible; (d) axistationary spacetimes with purely magnetic or electric Weyl tensor must be of Petrov type D.[1]

As a consequence of the above we can state the following: *A physically realistic rotating incompressible axistationary perfect fluid star model must be of Petrov type I.* Since it can be expected that the simplest physical axistationary perfect fluid spacetime, that can represent an astrophysical object with compact support, will be the incompressible ditto (compare with the interior Schwarzschild spacetime), it is reasonable to assume that algebraic restrictions on the Weyl tensor will not be fruitful in the search for such solutions.

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